Fair Allocation Visualizations

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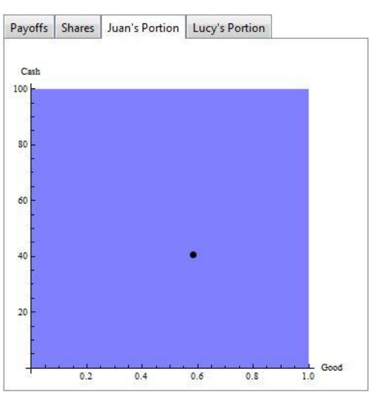
Abstract. Two or three people have equal ownership rights for several goods for which each person may have different monetary valuations. What is a fair way to allocate the goods among the people? This work examines different notions of fairness including efficient, envy-free, share equitable, and value equitable. The goal is to develop visualizations to assist us and others to better understand notions of fairness and their interrelationships.

1. Introduction

The goal is to generate interactive applets in Mathematica that visually display specific concepts of fairness. This paper looks at the payoff graphs for two and three people with a fixed amount of money and one or more goods. It also looks at the allocation graphs for the situation with two people, money, and one good. Each good is assumed to be divisible, and personal valuations are assumed to be additive and proportionate. This means that the value a person has for multiple goods equals the sum of values for its parts, and the value of each part is proportionate to the amount of the part. All allocations must consist only of goods and money in the system with each person's possible allocation ranging from zero to 100% of the good.

2. Feasible Region

Before one can look for fair allocations, it is nice to have an idea of all the allocations. The amount of money and the percent of every good that each person gets is their allocation. Suppose there are $m \geq 2$ people, $n \geq 1$ goods, and μ dollars. The feasible region represents all of the possible allocations to an inheritance problem. In an allocation graph for a single person j, money and each good i represents an axis. The feasible region is the hyperrectangle that ranges from 0 to 1 in each dimension, referencing the percentage of that good, and from 0 to μ for money.

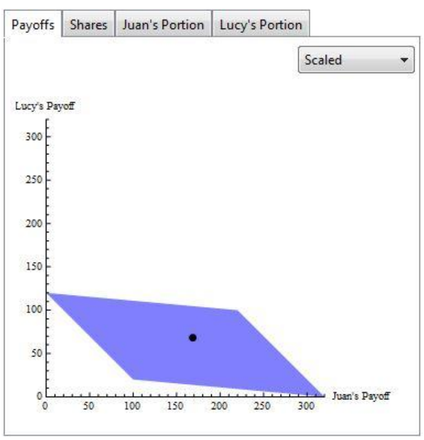


1. Allocation graph feasible region for one good and money.

Let x_{ij} be the allocation of good *i* to person *j* where $x_{ij} \geq 0$, and let d_j be the allocation of money to person *j* where $d_j \geq 0$, then $\sum_{j=1}^{m} x_{ij} = 1$ and $\sum_{j=1}^{m} d_j = \mu$. Since the combined allocations of a good cannot be more than 100%, everybody's allocations are interdependent.

Allocation graphs become very impractical when looking at money and more than two goods since there is no good way of viewing more than three dimensions. A payoff graph, and similarly a graph of each person's shares, allows the situation to be generalized to n goods without adding dimensions. These graphs are still limited to a maximum of three people, as each person represents an added dimension. Since the graph of each person's shares is the same concept as the payoff graph, where values are in terms of percentages instead of dollars, this paper will just focus on the payoff graph.

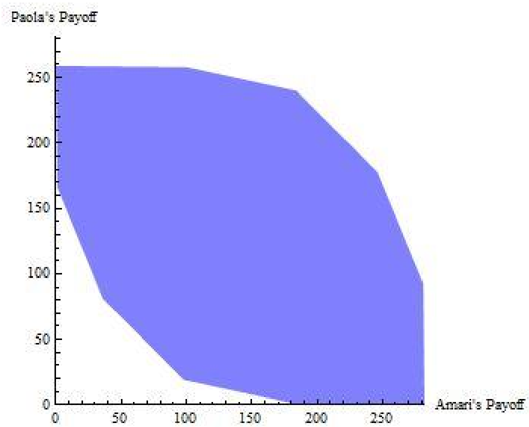
A payoff graph represents the total dollar amount each person receives based on their valuations and allocations. To get the feasible region in a payoff graph, every possible allocation has to be totaled based on each person's values. Let a_{ij} be the value of good *i* to person *j*, then the transformation from the allocation graph to the payoff graph can be represented by $(d_1 + \sum_{i=1}^{n} a_{i1} x_{i1}, d_2 + \sum_{i=1}^{n} a_{i2} x_{i2}, \ldots, d_m + \sum_{i=1}^{n} a_{im} x_{im})$.



2. Two person payoff graph feasible region for one good and money.

In order for Mathematica to draw this region, the programmer must provide it with the vertices of the region. In the situation where there was money and one good, the linear transformation from the allocation graph to the payoff graph was simple since every vertex in the allocation space was a vertex in the payoff space. As the goods increase in number, the transformation is occurring from a 2n-dimensional graph to a 2-dimensional graph. This means that some of the allocation vertices will be mapped inside of other vertices in the payoff region. To draw this region when there is money and more than one good, the convex hull of the vertices mapped to the payoff region must be found. A convex hull of a set of points is the smallest region that bounds all the points where there are no concave sections on the boundary.

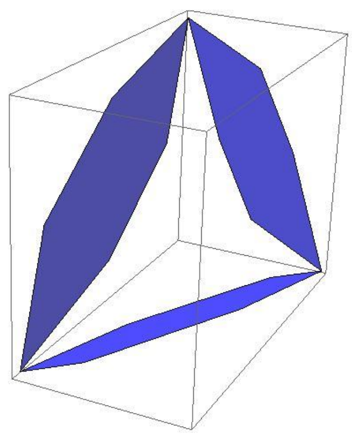
When dividing an inheritance between two people, one can take advantage of this specific situation to easily find the convex hull. When using the ratios between the two people's valuations, there is an algorithm which can return the vertices of the convex hull. Since the vertices of allocation graphs are where an individual receives all of a good or money, this algorithm, which does not divide the goods, will generate a subset of the allocation vertices. The first step is to give all the goods to one person. Then to determine how much that person values the goods compared to the other person, divide the first person's valuations by the second person's for every good. These ratios can be used to order the goods in order from smallest ratio to largest. To generate the rest of the vertices transfer the goods to the second person in order from smallest ratio to largest and then give them back to the first person in the same order. By transferring them in this order, the second person gets the greatest payoff increase per payoff loss for the first person. This will generate all the vertices with the highest payoffs, all the boundary vertices on the right half of the payoff graph. The two points where one person has all the goods are on the axis, since the other person has a payoff of zero. When transferring the goods back to the original person, each person values what the other person has the most. This generates the vertices with the lowest payoffs, meaning they are the vertices on the extreme left of the graph. This algorithm allows a quickly computable solution for the feasible region of two people and *n* goods without requiring use of the allocation graphs.



3. Two person payoff graph feasible region for multiple goods and money.

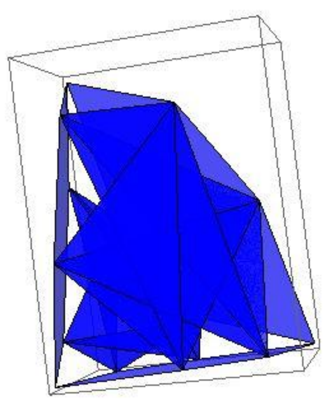
When dealing with three people, an extension of the algorithm for two players was not found. To select the outside points from the allocation vertices, a general purpose convex hull algorithm must be used to find the convex hull of the allocation vertices in three-dimensional payoff space. One possible algorithm is to start with a small region and then check each point against all the faces to see if it is included in the region. If the point is outside the region, the faces that it is outside of are deleted and that point is included as a vertex. This process is repeated until all of the points have been tested.

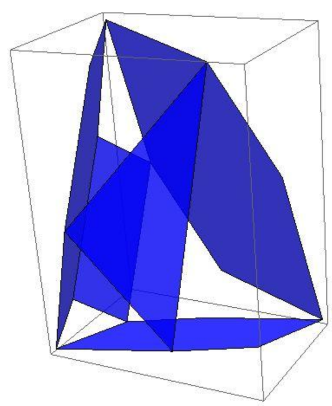
To reduce computer computation time, it is important to have the initial region to be as close to the final region as possible. This is especially important since there are 2^{n+1} points that need to be checked (the n+1 represents the number of goods plus money). This initial region can be found using knowledge from the 2D regions. Each plane determined by two of the axes contains all the possibilities where one person gets nothing. These problems can be solved using the same algorithm used for the two person problem.



4. Two person cross sections of three person payoff graph.

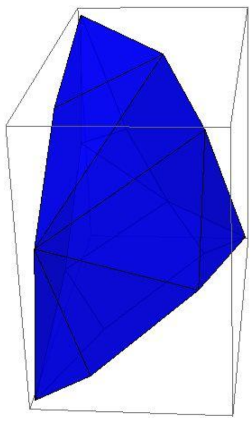
The final step of the initial convex region is to put top and bottom faces on the region bounded in image four. One algorithm requires that one first find the highest and lowest face on the region that spans all three boundaries. This is done by taking all the possible top and bottom faces and their normals, then comparing them by checking if the dot product between the normal and a point on the plane is greater than the dot product of the normal and the point in question (figure 5). The planes that are farthest out are merged and represent the central top and bottom plane (figure 6).





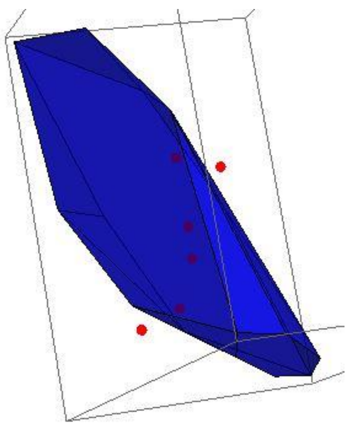
5. Potential extreme top and bottom faces. **6.** Extreme top and bottom faces.

These two faces are used to find the faces that cover the six regions between the apex vertices and the center faces. Starting at the apex, separately for the top and bottom, the apex plus one point along each side form a triangle. The next step is to move one direction along both boundary faces. Each of these two points would create a new face if connected with the two closest points on each boundary from the last plane. These two new potential faces are compared to see which one is above the other point. It is important to always go from “highest” face to the “lowest” to ensure there are no concave sections on the region. The “highest” face is checked for coplanarity and then either made into a new face or merged with the last one. This process is repeated until the center piece is reached. At this point the vertices and their connections have been found for an initial region that is similar to the final region.



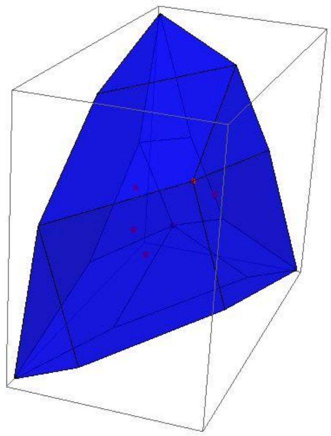
7. Initial convex hull.

From this point the general purpose algorithm can be used. It is only necessary to compute the vertices where each person has received something, as all other vertices are on the boundary.



8. Remaining unprocessed points.

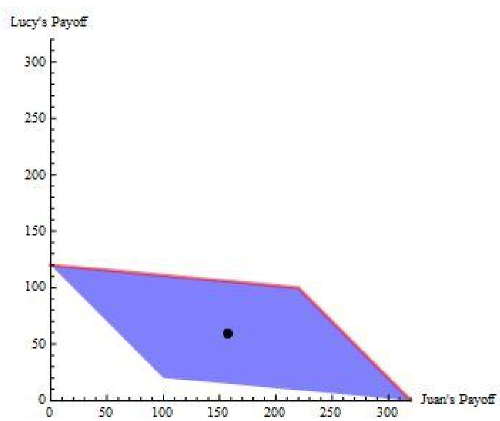
Each point is checked against the top and bottom faces. All the faces that are included by the point are deleted. These faces must all be connected since, by nature of a convex hull, there can be no intermediate face that is outside the point. By returning the sides of the deleted faces that are not bordered by other deleted faces, we obtain the boundary of the region inside of the point. Each of these sides adds the new point to make a new face. After checking to make sure none of these faces are coplanar, the initial region has added a new point. This process continues until all the points are contained within the convex hull.



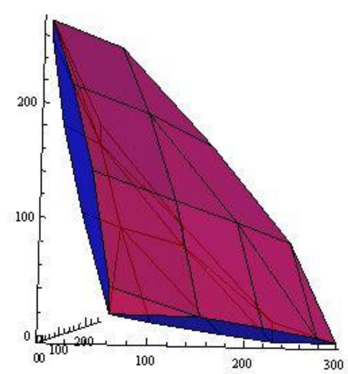
9. Final Convex Hull

3. Efficient

An allocation (*x*, *d*) is efficient if there is no other allocation (*y*, *e*) where $\mathrm{d} + \sum_{i=1}^{n} a_{ij} x \le e + \sum_{i=1}^{n} a_{ij} y$ for all *j* and $\mathrm{d} + \sum_{i=1}^{n} a_{ij} x < e + \sum_{i=1}^{n} a_{ij} y$ for at least one *j*. In other words, there is not a single division where one person is better off without harming one other person. In payoff space it would be all the points on the feasible region where it would be impossible to increase one value without decreasing another value. This would be the right edge of the two person feasible region and the top faces of the three person feasible region.



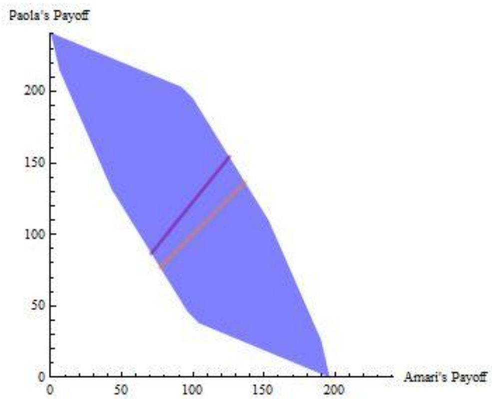
10. Two person efficient region.



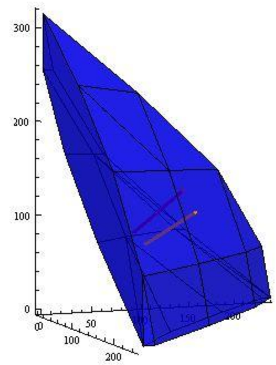
11. Three person efficient region.

4. Equitable

Equability is the set of solutions where every person receives the same size inheritance as every other person. There are two types of equitable solutions, the value equitable solution and the share equitable solution. A value equitable solution is an allocation where the payoff value for every person is equal: for all *j* and *k*, $d_j + \sum_{i=1}^{n} a_{ij} x_{ij} = d_k + \sum_{i=1}^{n} a_{ik} x_{ik}$. The share equitable solution is an allocation where every person gets the same percentage of the inheritance: for all *j* and k, $\frac{d_j+\sum_{i=1}^{n} a_{ij}x_{ij}}{d+\sum_{i=1}^{n} a_{ij}} = \frac{d_k+\sum_{i=1}^{n} a_{ik}x_{ik}}{d+\sum_{i=1}^{n} a_{ik}}$. This form of equability takes into account how much each person values the entire inheritance. For both the three dimensional and two dimensional graphs the solution set is a line segment. For equal value the line segment is contained within the feasible region and $y = x = z$, where x,y, and z are dimensions within the payoff space. For equal shares the relationship between the two or three dimensions is dependent on how much each person values the inheritance. The line segment would be similar to the one for equal values but instead of $y = x = z$, $\frac{y}{d+\sum_{i=1}^{n} a_{ij}} = \frac{x}{d+\sum_{i=1}^{n} a_{ik}} = \frac{z}{d+\sum_{i=1}^{n} a_{il}}$ for individuals j,k, and l.



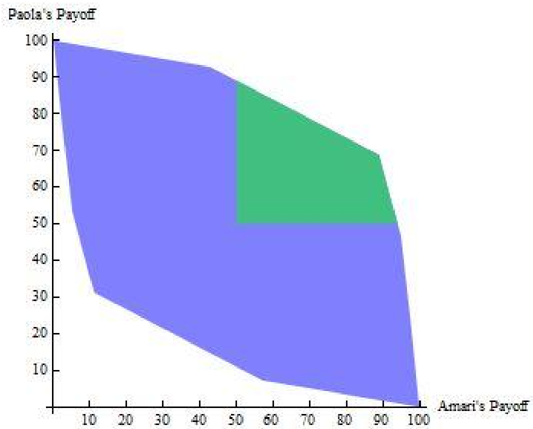
12. Two person equitable solutions. The purple is share equitable and the orange is value equitable.



13. Three person solutions. The purple is share equitable and the orange is value equitable.

5. Envy Free

The set of solutions in which each person values their share of the inheritance as equal to the best share, is the set of envy free solutions: for any two people j and k, $d_j + \sum_{i=1}^{n} a_{ij}x_{ij} \geq d_k + \sum_{i=1}^{n} a_{ij}x_{ik}$. This definition presents an easy way to identify the envy free solutions on a payoff graph. When there are only two people, it is possible to calculate the second person's share in relation to the first persons share. For example, if one person thinks he got 60% of the inheritance, then he thinks the other person received 40% of the inheritance. Therefore only solutions where each person receives 50% or more of their total valuation of the inheritance is envy free. If each person gets at least 50% of their maximum payoff, then the other persons share will be perceived as 50% or less of the total inheritance. With three people it is impossible to deduce every person's shares from a single person's share. This means that there is no minimum percentage of the inheritance for each person that ensures that all solutions with greater than or equal payoffs will be all the envy free solutions. If each person has at least 50% of their total valuation then all the solutions will be envy free, but that is not the complete set of envy free solutions. One person can have 40% of their total valuation as long as the other two people's shares are perceived as less than 40%. For the range between $\frac{1}{3}$ and $\frac{1}{2}$ of a persons valuations, the property of envy free is dependent on how the remaining goods are distributed among the other two people.



14. Envy free region on a two person payoff graph.

6. Conclusion

There are many different fair division situations and applications. This particular study focused on the single situation of dividing up an inheritance. To further simplify it, it only considered the two cases of two and three people. This study also does not attempt to analyze any methods of achieving the results it reports. Despite the limited scope and many assumptions made about people's values, this problem quickly becomes very complex to display graphically. This study looks at the most basic scenario's to help teach the different elements of fairness. While it would be interesting to extend these applets to other situations, they serve to teach the general concepts in a visual way to help others solve their division problems.